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Heat Losses from an Insulated Pipe

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Should domestic hot water pipes be thermally lagged? Such insulation may increase the thermal capacity of the pipe-insulation system to the point where, under intermittent use, more energy is lost with insulation around the pipes than without it. In this paper a model is presented for the insulated system and solutions derived to show the dependence of the energy losses on the properties of the system.

MODEL FOR THE SYSTEM

An infinitely long pipe of radius a takes fluid from a hot source at temperature T_1 and position $x = 0$, and transports it along the positive x-axis. The fluid at each cross-section is assumed to be well mixed, with a temperature at position x and time t given by $T(x, t)$. This condition is probably satisfied if the flow is turbulent.

The heat flux down the pipe is assumed to be $\pi a^2 V\rho cT - \beta(\partial T/\partial x)$, where V is the mean fluid velocity defined as volume flux per unit area, and ρcT is the thermal energy content of the fluid per unit volume. The term $-\beta(\partial T/\partial x)$ is the contribution due to conduction and dispersion along the pipe. If αT is the thermal energy of unit length of pipe when empty, then

$$\frac{\partial}{\partial t}((\pi a^2\rho c + \alpha)T) + \frac{\partial}{\partial x}\left[\pi a^2 V\rho cT - \beta\frac{\partial T}{\partial x}\right] = -H, \tag{1}$$

where H is the heat loss per unit length, through the surface $r = a$. Boundary and initial conditions require $T = T_1$ when $x = 0$, and $T = T_0$ at $t = 0$ for $x > 0$.

Suppose the pipe has thermal insulation of conductivity k_i, density ρ_i, and heat capacity per unit mass c_i. Moreover, suppose that at the outer edge $r = b$ of the insulation, which is at a temperature T_2, heat loss to the environment at temperature T_0 satisfies the linear radiation condition,

$$\frac{\partial T}{\partial r} = -\frac{\gamma}{k_i}(T_2 - T_0) \qquad (\text{at } r = b).$$

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Then, in the insulation, T satisfies the equations

$$\frac{k_i}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} = \rho_i c_i \frac{\partial T}{\partial t}, \qquad a < r < b,$$

on the outer surface of the insulation,

$$\frac{\partial}{\partial r}(T - T_0) = -\frac{\gamma}{k_i}(T - T_0) \qquad (\text{at } r = b), \tag{2}$$

and finally, T is constrained by the boundary conditions $T = T_1$ when $x = 0$, $T = T_0$ at $t = 0$ for $x > 0$.

At $r = a^+$, the heat loss per unit length of pipe is

$$H = -2\pi a k_i \frac{\partial T}{\partial r}, \tag{3}$$

which, with Eqs. (1) and (2), define the temperature profile.

In the case of V constant, our system of equations can be put in a dimensionless form by defining

$$\begin{aligned}
&x = lX, \qquad r = aS, && T = T_0 + \theta(T_1 - T_0),\\
&t = \left(\frac{l}{V} + \frac{al}{\pi a^2 V\rho c}\right)\tau, && \beta^* = \frac{\beta}{\pi a^2 V\rho c l},\\
&k^* = \frac{2k_i l}{a^2 V\rho c}, && \lambda^* = \frac{\pi a^4 \rho_i c_i V\rho c}{l k_i(\pi a^2 \rho c + \alpha)}\\
&\gamma^* = \frac{a\gamma}{k_i}, && S^* = \frac{b}{a}.
\end{aligned} \tag{4}$$

Then:

$$\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial X} = \beta^* \frac{\partial^2\theta}{\partial X^2} + k^* \left.\frac{\partial\theta}{\partial S}\right|_{1^+}; \qquad S \leqslant 1, X > 0, \tau > 0; \tag{5a}$$

$$\frac{1}{S}\frac{\partial}{\partial S} S \frac{\partial\theta}{\partial S} = \lambda^* \frac{\partial\theta}{\partial\tau}; \qquad 1 < S < S^*, X > 0, \tau > 0; \tag{5b}$$

$$\frac{\partial\theta}{\partial S} = -\gamma^*\theta; \qquad S = S^*, X > 0, \tau > 0;$$

$$\theta = 0 \quad \text{at} \quad \tau = 0, \quad X > 0;$$

$$\theta = 1, \quad X = 0, \quad S = 1, \quad \tau > 0.$$

A quantity of immediate interest is the accumulated energy loss, $E(x, t)$, defined as the thermal energy input at $x = 0$ up to time t, minus the output at x during the same time, so that

$$
\begin{aligned}
E(x, t) &= \int_0^t \pi a^2 V \rho c (T_1 - T)\, dt' \\
&= E_0 \int_0^\tau (1 - \theta)\, d\tau' \qquad (6) \\
E_0 &= (T_1 - T_0)(\alpha + \pi a^2 \rho c) l.
\end{aligned}
$$

Steady-State Solution

If V is constant for $t > 0$, then, as τ tends to infinity, the function θ tends to a steady solution θ_∞, which satisfies the system of equations,

$$
\begin{aligned}
&\frac{\partial \theta_\infty}{\partial X} = \beta^* \frac{\partial^2 \theta_\infty}{\partial X^2} + k^* \left.\frac{\partial \theta_\infty}{\partial S}\right|_{1^+}, && S \leqslant 1, X > 0; \\
&\frac{1}{S} \frac{\partial}{\partial S} S \frac{\partial \theta_\infty}{\partial S} = 0, && 1 < S < S^*, X > 0; \qquad (7) \\
&\frac{\partial \theta_\infty}{\partial S} = -\gamma^* \theta_\infty, && S = S^*, X > 0; \\
&\theta_\infty = 1, && X = 0.
\end{aligned}
$$

Also, the rate of loss of energy $(\partial/\partial t)\, E(x, t)$ at x and time t tends to the limiting value $E_0(1 - \theta_\infty)$. The solution to Eqs. (7) is

$$
\begin{aligned}
\theta_\infty &= (1 - \delta \log S) \exp(-\varepsilon X), \qquad 1 \leqslant S < S^*, \\
\delta &= 1 \Big/ \left(\log S^* + \frac{1}{S^* \gamma^*} \right), \qquad (8) \\
\varepsilon &= ((1 + 4k^* \delta \beta^*)^{1/2} - 1)/2\beta^*.
\end{aligned}
$$

This is required in the next section.

From Eqs. (8), the steady-state fluid temperature at position X is $\theta_\infty = \exp(-\varepsilon X)$, and so θ_∞ is maximal (minimal) when ε is minimal (maximal), or equivalently, when δ is minimal (maximal), provided γ^*, k^* and β^* are held fixed. From the dependence of δ on S^*, we see that θ_∞ is maximised as S^* tends to infinity. On the other hand, if γ^* is less than unity, then minimal θ_∞ occurs when $S^* = \gamma^{*-1}$ [1], while if $(\gamma^* > 1)$, $\min \theta_\infty = 1$.

Asymptotic Behaviour of Energy Loss

We can write the expression defining E in the form

$$E = E_0 \int_0^\tau (1 - \theta_\infty + \theta_\infty - \theta)\, d\tau,$$

$$= E_0(1 - \theta_\infty)\tau + E_0 \int_0^\tau (\theta_\infty - \theta)\, d\tau.$$

As τ tends to infinity, the function ϕ [2]

$$\phi = \int_0^\tau (\theta_\infty - \theta)\, d\tau,$$

approaches a limiting value ϕ_∞ which satisfies a boundary value problem derived from the system of equations satisfied by θ_∞ and θ. We see that

$$\beta^* \frac{\partial^2\phi}{\partial X^2} - \frac{\partial\phi}{\partial X} + k^* \frac{\partial\phi}{\partial S}\bigg|_{1^+} = \int_0^\tau -\frac{\partial\theta}{\partial\tau}\cdot d\tau = -\theta(X, \tau) \quad \text{on } S = 1, \qquad X > 0;$$

$$\frac{1}{S}\frac{\partial}{\partial S} S \frac{\partial\phi}{\partial S} = -\lambda^*\theta(X, \tau), \qquad 1 < S < S^*, \quad X > 0;$$

$$\frac{\partial\phi}{\partial S} + \gamma^*\phi = 0, \qquad \text{on } S = S^*, \quad X > 0;$$

$$\phi = \int_0^\tau (1 - 1)\, d\tau' = 0 \qquad \text{on } X = 0, \quad S = 1,$$

and so, letting τ tend to infinity, we obtain

$$\begin{aligned}
&\beta^* \frac{\partial^2\phi_\infty}{\partial X^2} - \frac{\partial\phi_\infty}{\partial X} + k^* \frac{\partial\phi_\infty}{\partial S}\bigg|_{1^+} = -\theta_\infty && \text{on } S = 1, && X > 0;\\
&\frac{1}{S}\frac{\partial}{\partial S} S \frac{\partial\phi_\infty}{\partial S} = -\lambda^*\theta_\infty && \text{on } 1 < S < S^*, && X > 0; \qquad (9)\\
&\frac{\partial\phi_\infty}{\partial S} + \gamma^*\phi_\infty = 0 && \text{on } S = S^*, && X > 0;\\
&\phi_\infty = 0 && \text{on } X = 0, && S = 1;\\
&\phi_\infty \to 0 && \text{on } S = 1 \quad \text{as} && X \to \infty,
\end{aligned}$$

where θ_∞ is given in Eq. (8).

A tedious calculation shows that the longitudinal equation for ϕ_∞ is

$$\beta^* \frac{\partial^2 \phi_\infty}{\partial X^2} - \frac{\partial \phi_\infty}{\partial X} - k^* \delta \phi_\infty = -\mu e^{-\epsilon X},$$

where

$$\mu = 1 + \frac{k^* \lambda^*}{4} \left(\frac{\delta^2}{\gamma^{*2}} [(S^* \gamma^*)^2 + 2S^* \gamma^* + 2] - [\delta^2 + 2\delta + 2] \right)$$

and so

$$\phi_\infty = \frac{\mu X e^{-\epsilon X}}{(1 + 2\beta^* \varepsilon)}.$$

We shall now summarise the results of this section. First, choose l, a normalisation length in Eqs. (4), so that $X = 1$. Then, for large time τ,

$$\begin{aligned} E(l, \tau)/E_0 &\simeq (1 - \theta_\infty)\tau + \phi_\infty \\ &= (1 - e^{-\epsilon})\tau + \frac{\mu e^{-\epsilon}}{1 + 2\beta^* \varepsilon}. \end{aligned} \tag{10}$$

The asymptotic formulae (Eq. (10)) gives an energy loss line for an unlagged pipe, and for one with $b - a$ of insulation. The intersection of these two lines give an estimate of the break-even time for the lagged case over the unlagged, and suggests that the pipe should be lagged if this time is significantly less than the mean time for hot water flows through the pipe.

The Transient Solutions

The previous section gives a good approximation in the long time limit. For small τ the energy loss E is approximately $E_0 \tau$, but the transition from this behaviour to the long time limit requires a more thorough discussion.

We derive in this section the complete solution to Eqs. (5).

(a) We begin by deriving a closed form solution of the problem with no insulation.

(b) We then include a layer of insulation, and show that the form of the Laplace transform of the solution is the same as that without insulation, but that now a numerical inversion of this transform is required to complete the solution.

(a) The equation for the transient temperature profile without insulation is

$$\left(\rho c + \frac{\alpha}{\pi a^2}\right)\frac{\partial T}{\partial t} + \rho c V \frac{\partial T}{\partial x} = \frac{\beta}{\pi a^2}\frac{\partial^2 T}{\partial x^2} - \frac{2\gamma}{a}(T - T_0), \tag{11}$$

$$T = T_0, \qquad x > 0, \qquad \tau = 0;\ T = T_1 \qquad \text{at} \quad x = 0, \tau > 0.$$

Equation (11) in dimensionless form is

$$\frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial X} = \beta^* \frac{\partial^2 \theta}{\partial X^2} - K\theta, \tag{12}$$

$$\theta = 0 \qquad \text{at} \quad \tau = 0, \qquad X > 0; \qquad \theta = 1 \qquad \text{at} \quad X = 0, \qquad \tau > 0,$$

where

$$K = \frac{2\gamma l}{a V \rho c}$$

and θ, X, τ, β^* have been defined in Eq (4).

From the Laplace transform definition

$$\bar{\theta} = \int_0^\infty e^{-p\tau}\theta\, d\tau$$

and Eq. (12), we obtain

$$\beta^* \frac{\partial^2 \bar{\theta}}{\partial X^2} - \frac{\partial \bar{\theta}}{\partial X} - (K + p)\bar{\theta} = 0,$$

which has as solution

$$\bar{\theta} = \frac{1}{p}\exp\left(-\frac{X}{2\beta^*}(1 + 4\beta^*(p + K))^{1/2}\right). \tag{13}$$

Consequently, the small time behaviour (large p) is independent of K. Furthermore, the maximal principle may be applied to Eq. (12) to show that the fluid temperature is bounded above by the solution corresponding to zero K.

The Inverse Laplace Transform of Eq (13) is [3]

$$\begin{aligned}\theta = \frac{1}{2}\exp\left(\frac{X}{2\beta^*}\right)\Bigg\{&\exp(-(K/\beta^* + 1/(2\beta^*)^2)^{1/2}\cdot X)\\ &\cdot \operatorname{erfc}\left[\left(\frac{X^2}{4\tau\beta^*}\right)^{1/2} - \left(\frac{\tau}{4\beta^*}\right)^{1/2} + K\tau\right] \\ &+ \exp\left((K/\beta^* + 1/(2\beta^*)^2)^{1/2}\right)\cdot \operatorname{erfc}\left[\left(\frac{X^2}{4\beta^*\tau}\right)^{1/2} + \left(\frac{\tau}{4\beta^*} + K\tau\right)^{1/2}\right]\Bigg\}.\end{aligned} \tag{14}$$

By noting that $\operatorname{erfc}(\infty) = 0$, and $\operatorname{erfc}(-\infty) = 2$, it is easy to show that as $\tau \to \infty$, the solution in Eq. (14) tends to the steady state solution in Eq. (8), provided we associate K with k^*.

(b) Finally, for completeness, we shall sketch the full solution for non-zero insulation. This involves solving the two coupled equations (5a) and (5b).

The Laplace Transform of Eq. (5b) is

$$\frac{\partial^2 \bar{\theta}_1}{\partial S^2} + \frac{1}{S}\frac{\partial \bar{\theta}_1}{\partial S} - p\lambda^* \bar{\theta}_1 = 0,$$

which has as solution

$$\bar{\theta}_1 = \alpha_1 I_0((p\lambda^*)^{1/2} S) + \alpha_2 K_0((p\lambda^*)^{1/2} S), \tag{15}$$

where α_1, α_2 are independent of S, I_0, K_0 are modified Bessel functions, and $\bar{\theta}_1$ is the Laplace Transform of θ in the insulation. The boundary conditions on $\bar{\theta}_1$ fix

$$\begin{aligned} \bar{\theta} &= \alpha_1 I_0(p\lambda^*)^{1/2} + \alpha_2 K_0(p\lambda^*)^{1/2}, \\ 0 &= \alpha_1[\gamma^* I_0((p\lambda^*)^{1/2} S^*) + (p\lambda^*)^{1/2} I_1((p\lambda^*)^{1/2} S^*)] \\ &\quad + \alpha_2[\gamma^* K_0((p\lambda^*)^{1/2} S^*) - (p\lambda^*)^{1/2} K_1((p\lambda^*)^{1/2} S^*)], \end{aligned} \tag{16}$$

where $\bar{\theta}$ is the Laplace Transform of the normalised temperature inside the pipe. Solving Eq. (16) for α_1 and α_2, and substituting the result into Eq. (15) shows that

$$k^* \left.\frac{\partial \bar{\theta}_1}{\partial S}\right|_{1^+} = -K^* \bar{\theta}, \tag{17}$$

where

$$\begin{aligned} K^* &= -k^*(p\lambda^*)^{1/2} \frac{\gamma_1 + \gamma_2}{\gamma_3 - \gamma_4}, \\ \gamma_1 &= I_1(p\lambda^*)^{1/2} [\gamma^* K_0((p\lambda^*)^{1/2} S^*) - (p\lambda^*)^{1/2} K_1((p\lambda^*)^{1/2} S^*)], \\ \gamma_2 &= K_1(p\lambda^*)^{1/2} [\gamma^* I_0((p\lambda^*)^{1/2} S^*) + (p\lambda^*)^{1/2} I_1((p\lambda^*)^{1/2} S^*)], \\ \gamma_3 &= I_0(p\lambda^*)^{1/2} [\gamma^* K_0((p\lambda^*)^{1/2} S^*) - (p\lambda^*)^{1/2} K_1((p\lambda^*)^{1/2} S^*)], \\ \gamma_4 &= K_0(p\lambda^*)^{1/2} [\gamma^* I_0((p\lambda^*)^{1/2} S^*) + (p\lambda^*)^{1/2} I_1((p\lambda^*)^{1/2} S^*)]. \end{aligned}$$

From Eq. (17) and the Laplace Transform of Eq. (5a),

$$\beta^* \frac{\partial^2 \bar{\theta}}{\partial X^2} - \frac{\partial \bar{\theta}}{\partial X} - (K^* + p)\bar{\theta} = 0,$$

which is of the same form as Eq. (12). Its solution can be found by replacing K by K^* in Eq. (13), and using a numerical technique to invert the Laplace Transform.

Conclusion

This paper contains a mathematical model for describing heat losses from lagged pipes. We restricted discussion to constant flows, and derived the steady state solution for lagged pipes, and the transient solution for unlagged pipes. In the general case of a transient lagged system, the Laplace Transform of the solution was presented. Numerical procedures are available for inverting this. Perhaps the most useful contribution is an asymptotic formula to the general solution. This gives an estimate of the time at which energy losses from a lagged pipe fall behind those from an unlagged pipe.

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